This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TCOMM.2017.2686865, IEEE Transactions on Communications

Capacity of Energy Harvesting Binary Symmetric Channels with a (σ, ρ)-Power Constraint

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***Abstract*—Capacity of energy harvesting communications with deterministic energy arrival and finite battery size is investigated. An abstraction of the physical layer is considered, where binary sequences are transmitted through a binary symmetric channel, and a cost function is associated with the transmission of each symbol. Upper and lower bounds on the channel capacity are derived for the general case by studying the normalized exponent of the cardinality of the set of feasible input sequences. Several upper bounds on the exponent are proposed by studying supersets of the feasible set. Lower bounds are derived by applying the binary entropy-power inequality and by using specific signaling schemes based on a save-and-transmit strategy. Numerical results are presented for several values of the energy arrival rate and battery size, validating the usefulness of the capacity bounds established for the energy harvesting channels.**

***Index Terms*—Capacity, energy harvesting communications, binary symmetric channels.**

I. Introduction

Using the harvested energy to transmit information is a promising feature of future communication networks, particularly in view of the myriads of smart devices and sensors envisioned in Internet of Things (IoT) [2]. As equipment can harvest energy from the environment, the lifetime of the network is prolonged and frequent battery replacement is avoided. In a typical sensor network of smart meters, each node senses environment parameters such as temperature and humidity. Upon collection, the measurements are uploaded to a fusion center via wireless links, where the computationally heavy tasks are performed. As such, most of the harvested energy is used for information transmission. Efficient transmission techniques are, therefore, of paramount importance, and attracted significant attention in recent years [3]. Given causality of energy harvesting and energy consumption, the energy harvesting communication system is not memoryless: Even with memoryless channels, the battery status constrains the set of symbols available for transmission in consecutive time slots. This poses challenges for the study of capacity of such systems.

The literature on energy harvesting communications can be divided in two branches. One branch revolves around the problem of *energy management* and the other focuses on the *capacity* of energy harvesting channel. In both cases, different assumptions on the energy arrival process and battery size have been advanced. Recent surveys can be found in [4], [5].

From the perspective of energy management, it is of interest to optimize the system in terms of throughput or efficiency-related metrics under the energy constraints posed by the random harvested energy and finite battery size. A generalization of water-filling, the so-called ‘directional water-filling’, is derived in [6] as the solution of the optimal power allocation problem over multiple time slots in order to maximize the throughput. Energy harvesting in connection with the additive white Gaussian noise (AWGN) channel was studied in the single user setting in [7], [8] and in the multiuser setting for broadcast channels in [9]. Energy cooperation through wireless energy transfer has been investigated in [10].

Most of the works in the branch of energy management assume generic nonnegative, increasing and concave rate-power relations. The results obtained through such analysis are upper bounds on the actual performance of energy harvesting systems. For example, if the expression for the capacity of an AWGN channel is used as the rate-power relation, the resulting throughput is an upper bound on the actual capacity of the energy harvesting AWGN channel [11], [12]. Therefore, it is of interest to study the actual information-theoretical capacity of energy harvesting channels.

From the perspective of capacity, the finite battery size assumption has proven to be a source of difficulties and a crucial component of the system. An early interesting result was that the capacity of the energy harvesting AWGN channel with infinite battery size is the same as the capacity of the AWGN channel with average power constraint [13]. This has been verified when the save-and-transmit strategy adopted in [13] was extended to the study of non-asymptotic achievable rates for energy-

This work is presented in part at the 2017 IEEE International Conference on Communications [1]. This work was supported in part by the SUTD-ZJU Research Collaboration under Grant SUTD-ZJU/RES/01/2014 and the MOE ARF Tier 2 under Grant MOE2015-T2-2-104.

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harvesting channels [14]. While an independent and identically distributed codebook is shown to be optimal by proving that the fraction of unfeasible codewords is vanishing [13], this is not the case when the battery size is finite. A first step towards understanding the finite battery case was made in [15], [16], where the communication channel was assumed noiseless. The problem is nontrivial due to the memory present in the sequence of transmitted symbols: A connection to a timing channel is instrumental in deriving a general capacity expression involving an auxiliary variable. Motivated by wireless signals propagation, this model was extended to include on-off fading in [17]. In [18], [19], capacity of the energy harvesting AWGN channel with finite battery was characterized up to a constant gap of approximately 2.58 bits. Slot-based and frame-based energy harvesting channels were compared in errorless case, where showed that the frame-based model can be emulated in the slot-based model [20]. Several contributions assume that the side information is available at the receiver side [21], [22] or regard the energy level in the battery as channel state, and bound capacity under different assumptions on the state knowledge available at the transmitter [23], [24].

Recently, the capacity of the energy harvesting AWGN channel with constant energy arrival and finite battery size was investigated [25]–[27]. In such channel, the arrival process is deterministic: ρ units of energy are harvested in each time slot. Furthermore, it is assumed that the battery can store at most σ units of energy. The constraint was referred to as (σ, ρ)-power constraint. In [25], lower and upper bounds on capacity were derived by using the entropy-power inequality and by assuming infinite battery, respectively. Tighter upper bounds were derived in [26] on the basis of a geometric analysis of the volume of the sum of the feasible set of input sequences and the noise Euclidean ball.

In this work, we study the capacity of an energy harvesting binary symmetric channel (EHBSC) with (σ, ρ)-power constraint. The setup is different from the literature [25]–[27], where a continuous variable is transmitted over an AWGN channel. We consider an abstraction of the physical layer where a binary symbol in $\{\mathtt{0}, \mathtt{1}\}$ is transmitted over a binary symmetric channel (BSC). A cost is associated with the transmission of each symbol, $\mathtt{0}$ being the zero-cost symbol. We investigate the capacity of the EHBSC by deriving bounds of the cardinality of the set of feasible input sequence (hereinafter referred to as *feasible set*).

The main contributions of this work are the following:

1) We establish upper bounds on the capacity of the general (σ, ρ)-power constrained EHBSC as a function of the normalized exponent of the cardinality of the feasible set (in Theorem 1). We propose upper bounds on the normalized exponent of the cardinality of the feasible set by finding the exponent for particular supersets of the feasible set (in Theorems 2-4), which are derived by removing part of the constraints posed by the energy arrival process on the set of input sequences.

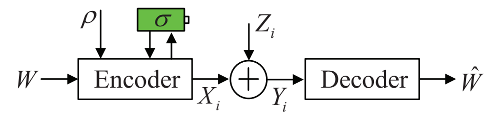

Fig. 1. Energy harvesting binary symmetric channel with (σ, ρ)-power constraint.

2) We establish lower bounds on the capacity of the general (σ, ρ)-power constrained EHBSC as a function of the normalized exponent of the cardinality of the feasible set by using the binary entropy-power inequality (in Theorem 1). We further provide lower bounds on the exponent by analyzing specific signaling schemes (in Theorems 5-7 and Corollary 1). These signaling schemes are based on a save-and-transmit strategy, where the transmitter is silent for a certain number of symbols to store energy in the battery and then transmits the codeword by using the stored energy.
3) We present numerical results for several values of σ and ρ, and show that the different bounds presented are close in different cases. In particular, we show that the bounds are close even in the case of small battery size which validates the usefulness of the developed bounds on the channel capacity.

The rest of this paper is organized as follows. In Section II, we introduce the EHBSC model. In Section III, we characterize the capacity of some special EHBSCs, and present bounds on the capacity for general cases as a function of the exponent of the cardinality of the feasible set. Section IV and Section V analyze the upper and the lower bounds on the exponent of the cardinality of the feasible set, respectively, which imply bounds on the capacity. Numerical results are shown in Section VI, and conclusion is drawn in Section VII.

Notation. Random variables are denoted by uppercase letters (e.g. X) and their realization by lowercase letters (e.g. x). Sets are denoted by calligraphic letters (e.g. $\mathcal{A}$) and their cardinality by $|\cdot|$ (e.g. $|\mathcal{A}|$). The shorthand $[a : b]$ is used to represent the set $\{a, a+1, \ldots, b\}$. A length-k vector of 1s is denoted by $\mathbf{1}_k$ and the $k \times k$ identity matrix is denoted by $\boldsymbol{I}_k$. The indicator function of set $\mathcal{X}$ is denoted $\mathbb{I}_{\mathcal{X}}$.

II. System model

Consider the energy harvesting communication channel illustrated in Fig. 1. The transmitter is equipped with a battery that can store up to σ units of energy. Time is slotted and in each time slot, a binary symbol $x \in \mathbb{Z}_2 := \{\mathtt{0}, \mathtt{1}\}$ is transmitted. A cost function $\mathsf{c} \colon \mathbb{Z}_2 \to [0, +\infty)$ is defined on $\mathbb{Z}_2$ to take into account the energy spent when a symbol is transmitted. In particular, we set without loss of generality $\mathsf{c}(\mathtt{1}) = 1$, i.e., the unit of energy is defined in terms of the energy spent to transmit the symbol $\mathtt{1}$. Furthermore, we assume that $\mathtt{0}$ is the zero-cost symbol, i.e., $\mathsf{c}(\mathtt{0}) = 0$. Hence, $\mathsf{c}(x) = x$. We assume the function c be separable, i.e., the cost of a sequence x^n is the sum of

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the costs of its symbols: $\mathsf{c}(x^n) = \sum_{i=1}^n x_i$. We note that the cost of the sequence is equal to its Hamming weight, i.e., the number of $\mathsf{1}$s in the sequence. Thus $\mathsf{c}(x^n) = \mathsf{w}(x^n)$ where $\mathsf{w}(x^n)$ denotes the Hamming weight of x^n. Denote σ_i the available energy in the battery at the beginning of time slot i, which is called *battery state*. In each time slot, $\rho \in [0, +\infty)$ units of energy are harvested. The transmitter charges its battery if the battery is not full. Transmitted symbols are constrained by the energy stored in the battery and the energy harvested in the time slot. By assuming that at time $i = 0$ the battery is full, the battery state evolves as

$$\sigma_{i+1} = \min\{\sigma_i + \rho - x_i, \sigma\}. \tag{1}$$

where $(x_1, x_2, \ldots)$ represents a feasible transmitted sequence, i.e., a sequence of symbols such that

$$\sigma_i \geqslant 0, \qquad i \geqslant 1. \tag{2}$$

Since σ_i depends on the symbol transmitted at time slot $(i-1)$, the battery state σ_i has memory. According to (2), the set of feasible input sequences is defined as follows:

$$\mathcal{S}_n(\sigma, \rho) := \{x^n \in \mathbb{Z}_2^n : \sigma_i \geqslant 0,\ 0 \leqslant i \leqslant n\}.$$

By telescoping the minimum operation in (1), one has

$$\sigma_i = \min\left(\sigma,\, \sigma + \rho - x_i,\, \ldots,\, \sigma + i\rho - \sum_{j=1}^{i} x_j\right) \geqslant 0.$$

Therefore, one can equivalently express the feasible set $\mathcal{S}_n(\sigma, \rho)$ as [28], [29]

$$\mathcal{S}_n(\sigma, \rho) = \Bigg\{x^n \in \mathbb{Z}_2^n : \sum_{j=i+1}^{i+l} x_j \leqslant \sigma + l\rho, \\ 0 \leqslant i \leqslant n-l,\ 1 \leqslant l \leqslant n\Bigg\}. \tag{3}$$

In words, the (σ, ρ)-power constraint selects those binary sequences having all subsequences with Hamming weight no larger than the total energy that can be spent during the subsequence, which is equal to the battery size plus the energy arrived during the subsequence.

Denote Y_i the received symbol and $Z_i \sim \mathrm{Bern}(q)$ the Bernoulli distributed noise with parameter q at time slot i. It leads to

$$Y_i = X_i + Z_i \mod 2. \tag{4}$$

We assume that $q \in [0, \frac{1}{2}]$ and the channel is memoryless, i.e., $P_{Z^n} = \prod_{i=1}^n P_Z$. A $(2^{\lfloor nR \rfloor}, n)$ code for the EHBSC with a (σ, ρ)-power constraint consists of a message set $[1 : 2^{\lfloor nR \rfloor}]$, an encoding function f: $[1 : 2^{\lfloor nR \rfloor}] \to \mathcal{S}_n(\sigma, \rho)$, and a decoding function g: $\mathbb{Z}_2^n \to [1 : 2^{\lfloor nR \rfloor}]$. If there exists a sequence of $(2^{\lfloor nR \rfloor}, n)$ codes such that the decoding error probability vanishes as $n \to \infty$, the rate R is said to be achievable. The capacity of the EHBSC with (σ, ρ)-power constraint, which is denoted by $C(\sigma, \rho)$, is defined as the supremum of all achievable rates.

III. Capacity Analysis

In III-A, we establish the capacity of the EHBSC in special cases. Then, we present upper and lower bounds on the capacity of the general EHBSC in III-B.

A. Special Cases

1) High Energy Arrival Rate: If $\rho \geqslant 1$, then in each time slot the transmitter can select $\mathsf{1}$ or $\mathsf{0}$ freely, irrespective of the battery size. In this case, the energy harvesting channel is equivalent to the BSC with cross probability q. The capacity is thus $1 - H_2(q)$, where $H_2(q) := -q \log_2 q - (1-q)\log_2(1-q)$ is the binary entropy function. Hence, $C(\sigma, \rho) = 1 - H_2(q)$ bits/use for $\rho \geqslant 1$.

2) Very Small Battery Size: If $\sigma < 1 - \rho$, then the transmitter cannot transmit symbol $\mathsf{1}$, irrespective of the time spent on harvesting energy. Thus, $C(\sigma, \rho) = 0$ for $\sigma + \rho < 1$. We refer to any battery size smaller than $1 - \rho$ as 'very small' battery size.

3) Infinite Battery Size: Consider a battery with infinite size, $\sigma = \infty$. In this case, the transmitter can store all the unused harvested energy into the battery for future transmissions. By closely following the save-and-transmit strategy proposed in [13], it can be shown that the capacity of the EHBSC with infinite battery is given by:

$$C(\infty, \rho) = \begin{cases} H_2(\rho * q) - H_2(q), & \text{if } \rho < \frac{1}{2}, \\ 1 - H_2(q), & \text{otherwise} \end{cases}, \tag{5}$$

where $\rho * q := \rho(1-q) + q(1-\rho)$. See also [20].

B. General Cases

Hereinafter, we study the EHBSC with (σ, ρ)-power constraint satisfying $\rho \in (0, 1)$ and $\sigma \geqslant 1 - \rho$. We refer to these EHBSC as *general* EHBSC. To study $C(\sigma, \rho)$, let us define the asymptotic normalized exponent of the cardinality of the feasible set

$$v(\sigma, \rho) := \lim_{n \to \infty} \frac{1}{n} \log_2 |\mathcal{S}_n(\sigma, \rho)|, \tag{6}$$

which represents the growth rate of the feasible set. The existence of $v(\sigma, \rho)$ follows from the sub-additivity of $\log_2 |\mathcal{S}_n(\sigma, \rho)|$. Specifically, consider a sequence $x^{n+n'} \in \mathcal{S}_{n+n'}(\sigma, \rho)$, then, $x^n \in \mathcal{S}_n(\sigma, \rho)$ and $x_{n+1}^{n+n'} \in \mathcal{S}_{n'}(\sigma, \rho)$. Therefore, $|\mathcal{S}_{n+n'}(\sigma, \rho)| \leqslant |\mathcal{S}_n(\sigma, \rho)||\mathcal{S}_{n'}(\sigma, \rho)|$.

We can bound the capacity of (σ, ρ)-power constrained general EHBSC as follows.

Theorem 1. *The capacity of (σ, ρ)-power constrained EHBSC, $C(\sigma, \rho)$, satisfies*

$$H_2\big(H_2^{-1}(v(\sigma, \rho)) * q\big) - H_2(q) \leqslant \\ C(\sigma, \rho) \leqslant \min\{C(\infty, \rho), v(\sigma, \rho)\}, \tag{7}$$

where q is the crossover probability of the BSC and $v(\sigma, \rho)$ is defined in (6).

Proof. Let $\mathcal{F}_n$ represents the set of all probability distributions supported almost surely on $\mathcal{S}_n(\sigma, \rho)$. Then, the

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channel capacity is given by [27, Theorem 10]

$$C(\sigma,\rho) = \lim_{n\to\infty} \frac{1}{n} \sup_{P_{X^n}\in\mathcal{F}_n} I(X^n;Y^n). \tag{8}$$

From the binary entropy-power inequality(Mrs. Gerber's Lemma) [30], it leads to

$$\frac{1}{n}H(Y^n) \geqslant H_2\left(H_2^{-1}\left(\frac{1}{n}H(X^n)\right) * q\right).$$

Hence, we have

$$\begin{aligned}\frac{1}{n}I(X^n;Y^n) =& \frac{1}{n}H(Y^n) - \frac{1}{n}H(Y^n|X^n) \\ \geqslant& H_2\left(H_2^{-1}\left(\frac{1}{n}H(X^n)\right) * q\right) - H_2(q).\end{aligned} \tag{9}$$

Note that both $H_2(p*q)$ and $H_2^{-1}(p)$ are increasing in p. Thus, the right hand side of (9) is increasing in $\frac{1}{n}H(X^n)$ and we have

$$\begin{aligned}&\sup_{P_{X^n}\in\mathcal{F}_n} \frac{1}{n}I(X^n;Y^n) \\ \geqslant& H_2\left(H_2^{-1}\left(\sup_{P_{X^n}\in\mathcal{F}_n}\frac{1}{n}H(X^n)\right) * q\right) - H_2(q) \\ \overset{\text{(a)}}{=}& H_2\left(H_2^{-1}\left(\frac{1}{n}\log_2|\mathcal{S}_n(\sigma,\rho)|\right) * q\right) - H_2(q).\end{aligned}$$

where (a) follows from the uniform distribution over $\mathcal{S}_n(\sigma,\rho)$ being optimum. By arguing the subadditivity of $\sup_{P_{X^n}\in\mathcal{F}_n}\frac{1}{n}I(X^n;Y^n)$ [27], one can guatantee the existence of $\lim_{n\to\infty}\sup_{P_{X^n}\in\mathcal{F}_n}\frac{1}{n}I(X^n;Y^n)$. As both $H_2(p*q)$ and $H_2^{-1}(p)$ are continuous in p, we have

$$\begin{aligned}&\lim_{n\to\infty}\sup_{P_{X^n}\in\mathcal{F}_n} \frac{1}{n}I(X^n;Y^n) \\ \geqslant& H_2\left(H_2^{-1}\left(\lim_{n\to\infty}\frac{1}{n}\log_2|\mathcal{S}_n(\sigma,\rho)|\right) * q\right) - H_2(q) \\ =& H_2\big(H_2^{-1}(v(\sigma,\rho)) * q\big) - H_2(q),\end{aligned} \tag{10}$$

To derive the upper bound, note that

$$\begin{aligned}\lim_{n\to\infty}\sup_{P_{X^n}\in\mathcal{F}_n} \frac{1}{n}I(X^n;Y^n) \leqslant& \lim_{n\to\infty}\sup_{P_{X^n}\in\mathcal{F}_n} \frac{1}{n}H(X^n) \\ =& \lim_{n\to\infty}\frac{\log_2|\mathcal{S}_n(\sigma,\rho)|}{n} \\ =& v(\sigma,\rho).\end{aligned} \tag{11}$$

Noting that $C(\sigma,\rho)$ is naturally bounded by $C(\infty,\rho)$ and combining (8), (10), and (11) yields the capacity bounds in (7). □

IV. Upper Bounds on $v(\sigma,\rho)$

In this section, we present upper bounds on the exponent $v(\sigma,\rho)$, which imply upper bounds on the channel capacity $C(\sigma,\rho)$.

To study $v(\sigma,\rho)$, let us relax some of the constraints that define the set $\mathcal{S}_n(\sigma,\rho)$. In particular, define the set

$$\mathcal{S}_n^{(l)}(\sigma,\rho) := \left\{x^n\in\mathbb{Z}_2^n : \sum_{j=i+1}^{i+l} x_j \leqslant \sigma + l\rho,\ 0\leqslant i\leqslant n-l\right\}. \tag{12}$$

For notational convenience, we shall remove the dependence on σ and ρ when there is no risk of confusion and denote

$$w_l := \sigma + l\rho \tag{13}$$

the constraint on the Hamming weight of any length-l subsequence of $x^n \in \mathcal{S}_n^{(l)}$, i.e.,

$$\mathsf{w}(x_{i+1}^{i+l}) \leqslant w_l, \quad i\in[0:n-l]. \tag{14}$$

We refer to the constraint (14) as (l, w_l)-constraint. The difference between (12) and (3) is that in (12) there is (l, w_l)-constraint only while in (3) there are all (l, w_l)-constraints for $l\in[1:n]$. Then, $\mathcal{S}_n = \bigcap_{l=1}^n \mathcal{S}_n^{(l)}$ and we can upper bound $v(\sigma,\rho)$ by studying $\mathcal{S}_n^{(l)}$ in place of $\mathcal{S}_n$ since

$$|\mathcal{S}_n| \leqslant |\mathcal{S}_n^{(l)}|, \quad l\in[1:n].$$

By evaluating $|\mathcal{S}_n^{(n)}|$, we have the following upper bound on $v(\sigma,\rho)$.

Theorem 2. *The exponent $v(\sigma,\rho)$ satisfies*

$$v(\sigma,\rho) \leqslant H_2\left(\min\left\{\rho, \frac{1}{2}\right\}\right). \tag{15}$$

Proof. Note that $|\mathcal{S}_n| \leqslant |\mathcal{S}_n^{(n)}|$. The sequences in $\mathcal{S}_n^{(n)}$ are binary sequences with Hamming weight no larger than $w_n = \sigma + n\rho$. Therefore,

$$|\mathcal{S}_n^{(n)}| = \sum_{i=0}^{\lfloor w_n\rfloor}\binom{n}{i}.$$

We shall evaluate an upper bound of $\frac{1}{n}\log_2|\mathcal{S}_n^{(n)}|$ as $n\to\infty$. Suppose $\rho < 1/2$. Since

$$\frac{\lfloor\sigma+n\rho\rfloor}{n} \leqslant \frac{\sigma+n\rho}{n} = \rho + \frac{\sigma}{n} < \frac{1}{2}$$

for $n > \sigma/(\frac{1}{2}-\rho) =: N$, it results in $\lfloor w_n\rfloor/n < 1/2$ for $n > N$. Therefore, for $n > N$,

$$\begin{aligned}|\mathcal{S}_n^{(n)}| &\leqslant (\lfloor w_n\rfloor + 1)\max_{0\leqslant i\leqslant\lfloor w_n\rfloor}\binom{n}{i} \\ &\leqslant (\lfloor w_n\rfloor + 1)\max_{0\leqslant i\leqslant\lfloor w_n\rfloor} 2^{nH_2(i/n)} \\ &= (\lfloor w_n\rfloor + 1)\, 2^{nH_2(\lfloor w_n\rfloor/n)}.\end{aligned}$$

Hence

$$\begin{aligned}v(\sigma,\rho) =& \lim_{n\to\infty}\frac{1}{n}\log_2|\mathcal{S}_n(\sigma,\rho)| \\ \leqslant& \lim_{n\to\infty}\frac{1}{n}\log_2|\mathcal{S}_n^{(n)}| \\ \leqslant& \lim_{n\to\infty} H_2(\lfloor w_n\rfloor/n) \\ =& H_2(\rho).\end{aligned}$$

It remains to prove that $v(\sigma,\rho) \leqslant H_2(1/2)$ when $\rho \geqslant 1/2$. Since $\mathcal{S}_n^{(n)}$ is a subset of $\mathbb{Z}_2^n$, $|\mathcal{S}_n^{(n)}|$ is bounded by $|\mathbb{Z}_2^n| = 2^n$. Accordingly, $v(\sigma,\rho)$ is upper bounded by $1 = H_2(1/2)$. □

It is noteworthy that the upper bound (15) relates to the corresponding infinite battery size noiseless channel capacity. In fact, the growth exponent of the number of feasible sequences, $v(\sigma,\rho)$, is exactly the capacity of the corresponding noiseless EHBSC with battery size σ, which can be upper bounded by the capacity of noiseless EHBSC with infinite battery. Therefore, (15) also can be obtained from setting the crossover probability $q=0$ in (5).

Note that the upper bound given in (15) does not depend on σ. Next, we derive bounds that depend on both σ and ρ. Since (l,w_l)-constraint is inactive if $l \leqslant w_l$, it is more valuable to focus on some subsequence length $l > w_l$. As $\sigma+\rho \geqslant 1$ and $\rho < 1$, for $j \in \mathbb{N}$, there always exists $l \in \mathbb{Z}$ satisfying

$$l - j \leqslant w_l = \sigma + l\rho < l - j + 1.$$

This is so because $\mathsf{w} = l-j$ and $\mathsf{w} = l-j+1$ are lines with slope equal to 1 while $\mathsf{w} = \sigma + l\rho$ is a line with slope equal to $\rho < 1$. Therefore, irrespective of σ, the intersections of $\mathsf{w} = \sigma + l\rho$ with $\mathsf{w} = l_j - j$ and $\mathsf{w} = l - j + 1$ are $l = (\sigma+j)/(1-\rho)$ and $l = (\sigma+j-1)/(1-\rho)$, respectively, and their difference is $1/(1-\rho) > 1$, hence there exists at least one integer between the two intersections. Let us denote the smallest integer corresponding to j by

$$l_j := \left\lfloor \frac{\sigma + j - 1}{1-\rho} \right\rfloor + 1. \qquad (16)$$

The particularity of l_j is that, for fixed σ and ρ, the Hamming weight of each length-l_j window in $\mathcal{S}_n^{(l_j)}$ is lower than $l_j - j + 1$, i.e.

$$\mathsf{w}(x_{i+1}^{i+l_j}) \leqslant w_j < l_j - j + 1, \quad i \in [0 : n - l_j].$$

In particular, $\mathsf{w}(x_{i+1}^{i+l_1}) \leqslant w_{l_1} < l_1$ for $i \in [0 : n - l_j]$, i.e., the number of the consecutive $\mathtt{1}$s in the input sequence should be less than l_1. Observing this, we derive the following upper bound on $v(\sigma,\rho)$ by considering specific window length $l = l_1$ and $l = l_2$.

Theorem 3. *The exponent of* $|\mathcal{S}_n(\sigma,\rho)|$ *satisfies*

$$v(\sigma,\rho) \leqslant \log_2 \min\{\beta_1, \beta_2\}, \qquad (17)$$

where $1/\beta_1$ *and* $1/\beta_2$ *are the zero of* $Q_1(z) = 1 - 2z + z^{l_1+1}$ *and that of* $Q_2(z) = 1 - 2z + (l_2-1)z^{l_2+1} - \sum_{j=1}^{l_2-1} z^{l_2+1+j}$ *with smallest modulus, respectively.*

Proof. See Appendix A. □

To establish a general upper bound using $\mathcal{S}_n^{(l)}$, we consider the partition of $\mathcal{S}_n^{(l)}$ in K disjoint subsets $\{\mathcal{S}_{n1}^{(l)}, \dots, \mathcal{S}_{nK}^{(l)}\}$, $\mathcal{S}_n^{(l)} = \amalg_{k=1}^{K} \mathcal{S}_{nk}^{(l)}$, where $\mathcal{S}_{nk}^{(l)}$ is the subset of sequences in $\mathcal{S}_n^{(l)}$ with the l trailing (rightmost) symbols representing $k-1$ in binary notation:

$$\mathcal{S}_{nk}^{(l)} := \left\{ x^n \in \mathcal{S}_n^{(l)} : (x_{n-l+1}^n)_{10} = k - 1 \right\}, \qquad (18)$$

where $(\cdot)_{10}$ denotes the operator that outputs the decimal value of the binary string in argument. In general, there can be $K := 2^l$ possible subsets, some of which can be empty because of the Hamming constraint (14). Let $z_{nk}^{(l)} := |\mathcal{S}_{nk}^{(l)}|$ be the cardinality of subset k and arrange these cardinalities into the vector $\boldsymbol{z}_n^{(l)} := (z_{n1}^{(l)}, \dots, z_{nK}^{(l)})^\mathsf{T}$.

We are interested in the above partition because the following relation between $\boldsymbol{z}_n^{(l)}$ and $\boldsymbol{z}_{n+1}^{(l)}$ holds:

$$\boldsymbol{z}_{n+1}^{(l)} = \boldsymbol{A}_l \boldsymbol{z}_n^{(l)}, \qquad (19)$$

where $(\boldsymbol{A}_l)_{ij} = 1$ if $\mathcal{S}_{ni}^{(l)} \neq \varnothing$ and each sequence $x^n \in \mathcal{S}_{ni}^{(l)}$ can generate a sequence $x^{n+1} \in \mathcal{S}_{(n+1)j}^{(l)}$ by padding a $\mathtt{0}$ or $\mathtt{1}$ at the position $n+1$, and $(\boldsymbol{A}_l)_{ij} = 0$ otherwise. Note that $\boldsymbol{A}_l \in \mathbb{R}^{K\times K}$ depends on l but does not depend on n. In the following theorem, we derive an upper bound on the exponent $v(\sigma,\rho)$ based on spectral properties of $\boldsymbol{A}_l$.

Theorem 4. *The exponent of* $|\mathcal{S}_n(\sigma,\rho)|$ *satisfies*

$$v(\sigma,\rho) \leqslant \min_{l\geqslant 1} \log_2 \lambda_{\max}(\boldsymbol{A}_l), \qquad (20)$$

where $\lambda_{\max}(\boldsymbol{A}_l)$ *is the maximum eigenvalue of* $\boldsymbol{A}_l$*.*

Proof. By iteratively applying (19) backwards we derive

$$\boldsymbol{z}_n^{(l)} = \boldsymbol{A}_l^{n-l} \boldsymbol{z}_l^{(l)},$$

Besides, by the partition of $\mathcal{S}_n^{(l)}$, we have

$$|\mathcal{S}_n^{(l)}| = \sum_{k=1}^{K} z_{nk}^{(l)} = \|\boldsymbol{z}_n^{(l)}\|_1,$$

being $z_{nk}^{(l)} \geqslant 0$. Therefore, we can express the cardinality of $\mathcal{S}_n^{(l)}$ in terms of $\boldsymbol{A}_l$ and $\boldsymbol{z}_l^{(l)}$:

$$|\mathcal{S}_n^{(l)}| = \|\boldsymbol{A}_l^{n-l} \boldsymbol{z}_l^{(l)}\|_1. \qquad (21)$$

Now we bound the ℓ_1-norm as follows:

$$\|\boldsymbol{A}_l^{n-l}\boldsymbol{z}_l^{(l)}\|_1 \leqslant \|\boldsymbol{A}_l^{n-l}\|_1 \|\boldsymbol{z}_l^{(l)}\|_1 \leqslant 2^l \|\boldsymbol{A}_l^{n-l}\|_1, \qquad (22)$$

where the matrix norm is induced by the ℓ_1-norm, the first inequality follows from the submultiplicativity of the norm, and the second inequality follows from the fact $\mathcal{S}_l^{(l)} \subseteq \mathbb{Z}_2^l$. Using the Gel'fand's spectral radius formula [31, page 349] yields

$$\lim_{n\to\infty} \|\boldsymbol{A}_l^{n-l}\boldsymbol{z}_l^{(l)}\|_1^{\frac{1}{n-l}} \leqslant \lim_{n\to\infty} 2^{\frac{l}{n-l}} \|\boldsymbol{A}_l^{n-l}\|_1^{\frac{1}{n-l}} = \lambda_{\max}(\boldsymbol{A}_l), \qquad (23)$$

being l fixed. From (21) and (23) we can upper bound the exponent of $|\mathcal{S}_n^{(l)}|$ as follows:

$$\begin{aligned} v(\sigma,\rho) &= \lim_{n\to\infty} \frac{1}{n} \log_2 |\mathcal{S}_n(\sigma,\rho)| \\ &\leqslant \lim_{n\to\infty} \frac{1}{n-l} \log_2 |\mathcal{S}_n^{(l)}| \\ &= \log_2 \lim_{n\to\infty} \|\boldsymbol{A}_l^{n-l}\boldsymbol{z}_l^{(l)}\|_1^{\frac{1}{n-l}} \\ &\leqslant \log_2 \lambda_{\max}(\boldsymbol{A}_l). \end{aligned}$$

Tightening the bound with respect to l yields the theorem. □

It is noteworthy that Theorem 4 can be regarded as an application of Perron-Frobenius theory in the context of constrained coding [32]. In fact, the set $\mathcal{S}_n^{(l)}$ can be described as a constrained system and presented by a finite

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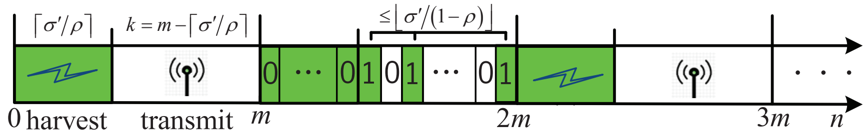

Fig. 2. The save-and-transmit strategy. The transmitter alternates harvesting and transmission phases of length $m-k$ and k symbols, respectively. Energy is harvested during the harvesting phase and during transmission of `1`s (filled slots). In the harvesting phase, `0`s are transmitted.

state graph [33].

In general, it is difficult to find the explicit expression of $\lambda_{\max}(\boldsymbol{A}_l)$. For specific $l=l_1$, we can derive $\lambda_{\max}(\boldsymbol{A}_{l_1})$ in a closed form which implies from Theorem 4 the following bound:

$$v(\sigma,\rho) \leqslant 1-\frac{2^{-l_1-1}}{\log 2}. \tag{24}$$

For fluency, the detail of the derivation of (24) is relegated to Appendix B.

V. Lower Bounds on $v(\sigma,\rho)$

In this section, we present lower bounds on $v(\sigma,\rho)$ which imply lower bounds on the channel capacity $C(\sigma,\rho)$.

Before specifying any bound, let us note that $v(\sigma,\rho)$ is nondecreasing as a function of both arguments. For fixed ρ, any bound valid for $\sigma=\sigma_1\leqslant\sigma_2$ is also valid for $\sigma=\sigma_2$ by using part of the battery than the whole battery. For fixed σ, any bound valid for $\rho=\rho_1\leqslant\rho_2$ is also valid for $\rho=\rho_2$ by harvesting less energy per time slot than the amount of energy that can be potentially harvested.

In the following theorem, we lower bound the cardinality of the feasible set by sticking to a specific transmission scheme based on the save-and-transmit strategy (see Fig. 2). According to this strategy, the transmitter alternates harvesting and transmission phases of length $m-k$ and k symbols, respectively. In the harvesting phase, the battery is charged to a preset level σ'. Then, in the transmission phase, k symbols are transmitted using the energy stored in the battery.

Theorem 5. *The exponent of $|\mathcal{S}_n(\sigma,\rho)|$ satisfies*

$$v(\sigma,\rho) \geqslant \max_{\substack{1-\rho\leqslant\sigma'\leqslant\sigma\\ k>0}} \frac{1}{k+\lceil\sigma'/\rho\rceil}\log_2\left[\sum_{i=0}^{i_{max}}\binom{k}{i}\right], \tag{25}$$

where $i_{max}=\min\{k,\lfloor\sigma'/(1-\rho)\rfloor\}$.

Proof. Suppose that the input sequence length n is divided into subsequences of length m. Each subsequence represents a harvest-and-transmit cycle. The harvesting phase lasts for $\lceil\sigma'/\rho\rceil$ symbols, therefore the transmission phase length is $k:=m-\lceil\sigma'/\rho\rceil$. During the transmission phase, the cost for the transmission of each symbol `1` is reduced from 1 to $1-\rho$ by using the energy arrived during the slot (cf. Fig. 2). Therefore, the number of `1`s that can be transmitted in the transmission phase is at most $i_{\max}:=\min\{k,\lfloor\sigma'/(1-\rho)\rfloor\}$. Accordingly, the number of sequences that we can transmit is $\sum_{i=0}^{i_{\max}}\binom{k}{i}$. Then, the cardinality of the feasible set is lower bounded as follows:

$$|\mathcal{S}_n(\sigma,\rho)| \geqslant \left[\sum_{i=0}^{i_{\max}}\binom{k}{i}\right]^{\lfloor n/m\rfloor}.$$

Therefore, $v(\sigma,\rho)$ can be bounded as follows:

$$\begin{aligned} v(\sigma,\rho) &= \lim_{n\to\infty}\frac{1}{n}\log_2|\mathcal{S}_n(\sigma,\rho)| \\ &\geqslant \lim_{n\to\infty}\frac{1}{n}\cdot\left\lfloor\frac{n}{m}\right\rfloor\log_2\left[\sum_{i=0}^{i_{\max}}\binom{k}{i}\right] \\ &= \frac{1}{k+\lceil\sigma'/\rho\rceil}\log_2\left[\sum_{i=0}^{i_{\max}}\binom{k}{i}\right]. \end{aligned} \tag{26}$$

The statement follows by tightening the bound with respect to $\sigma'\in[1-\rho,\sigma]$ and $k>0$. □

Using lower bound on the binomial coefficient, it is possible to further lower bound (25) as in the following corollary.

Corollary 1. *The exponent of $|\mathcal{S}_n(\sigma,\rho)|$ satisfies*

$$v(\sigma,\rho) \geqslant \max_{\substack{1-\rho\leqslant\sigma'\leqslant\sigma\\ k>0}} \frac{1}{k+\lceil\sigma'/\rho\rceil}\left[kH_2(\alpha)-\Pi\right], \tag{27}$$

where $\alpha:=\min\{\lfloor\sigma'/(1-\rho)\rfloor/k,\lfloor k/2\rfloor/k\}$ *and* Π *is a penalty term given by*

$$\Pi := \frac{1}{2}\log_2\frac{1}{\alpha(1-\alpha)k}+\frac{1}{2}\log_2(2\pi)+\frac{1}{8k}\log_2 e.$$

Proof. We lower bound $\sum_{i=0}^{i_{\max}}\binom{k}{i}$ in (25) as follows:

$$\sum_{i=0}^{i_{\max}}\binom{k}{i} = \sum_{i=0}^{\min\{k,\lfloor\sigma'/(1-\rho)\rfloor\}}\binom{k}{i} \geqslant \binom{k}{\alpha k}. \tag{28}$$

In words, if $\lfloor\sigma'/(1-\rho)\rfloor$ is larger than $\lfloor k/2\rfloor$, then we choose the maximum in $\left\{\binom{k}{i}:0\leqslant i\leqslant\lfloor\sigma'/(1-\rho)\rfloor\right\}$, which is $\binom{k}{\lfloor k/2\rfloor}$; otherwise, we choose the last term in the summation because the sequence $\binom{k}{i}$ is increasing as a function of i for $i\leqslant\lfloor k/2\rfloor$. Now we apply a standard lower bound on the binomial coefficient [34, Theorem 2.5], namely:

$$\binom{k}{\alpha k} \geqslant \frac{1}{\sqrt{2\pi}}2^{kH_2(\alpha)}k^{-1/2}\alpha^{-1/2}(1-\alpha)^{-1/2}e^{-\frac{1}{8k}}. \tag{29}$$

Combining (25), (28) and (29) result in the lower bound given in the statement. □

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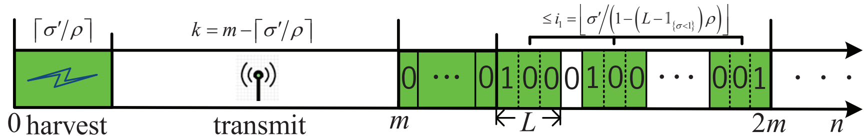

Fig. 3. An enhanced save-and-transmit strategy, which employs length-L, unit-weight supersymbols.

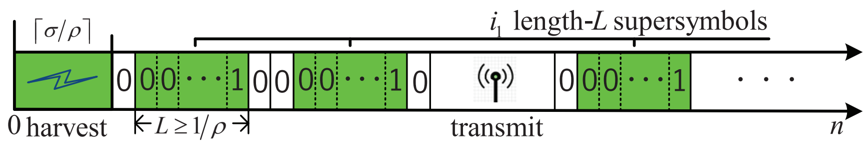

Fig. 4. The third variant of the save-and-transmit strategy, which employs length-L supersymbol $\texttt{00}\cdots\texttt{01}$ with $L\rho \geqslant 1$.

In an enhanced save-and-transmit strategy, we can constrain the transmitter to send either $\texttt{0}$ or a length-L unit-weight subsequence during which the transmitter always harvests energy (cf. Fig. 3). We can think of the length-L subsequence as a 'supersymbol'. The particularity of this scheme is that, by choosing $L < 1/\rho$, the difference between the energy in the battery after and before the transmission of the supersymbol is nonnegative. Hence, the system can transmit more $\texttt{1}$s in the transmission phase and there are more feasible sequences used in the transmission phase. In the below theorem, we state a bound on $v(\sigma,\rho)$ obtained by following this scheme.

Theorem 6. *Fix $\rho < 1/2$. The exponent of $|\mathcal{S}_n(\sigma,\rho)|$ satisfies*

$$v(\sigma,\rho) \geqslant \max_{\substack{1-\rho\leqslant\sigma'\leqslant\sigma\\ 0<L<1/\rho\\ k>0}} \frac{1}{k+\lceil\sigma'/\rho\rceil}\log_2\left[\sum_{i=0}^{i_{\max}}\binom{k-(L-1)i}{i}\right], \tag{30}$$

where $i_{\max} = \min\{k/L, \lfloor\sigma'/(1-(L-\mathbb{I}_{\{\sigma<1\}})\rho)\rfloor\}$.

Proof. Let i_0 and i_1 be the number of symbol $\texttt{0}$ and length-L supersymbols transmitted. Then $i_0 + Li_1 = k$. There are L possible supersymbols S_j, $j \in [1:L]$, that can be indexed by the position of the transmitted $\texttt{1}$. The signaling scheme is as follows: The transmitter chooses the supersymbol with lowest index that can be transmitted with the available energy in the battery. Note that the battery level after and before transmission of a supersymbol is always decreasing: this implies a bijection between the set of binary sequences that can be transmitted and a sequence of $\texttt{0}$ and supersymbols. Hence, the number of sequences that we can transmit is

$$\binom{i_0+i_1}{i_1} = \binom{k-(L-1)i_1}{i_1}.$$

Once $i_1 \leqslant i_{\max}$ is proved, (30) follows along a similar argument to that in the proof of Theorem 5. What remains is to show that $i_1 \leqslant i_{\max}$. First, $k-(L-1)i_1$ should be not less than i_1, hence, $i_1 \leqslant k/L$. Let the battery level before transmission of a supersymbol be $\bar{\sigma}$. It is always true that $\bar{\sigma} \geqslant 1 - L\rho$, otherwise it is impossible to transmit a supersymbol. If $\bar{\sigma} \geqslant 1-\rho$, then we transmit S_1. Otherwise, $\bar{\sigma} < 1-\rho$ and we assume that $1 - j\rho \leqslant \bar{\sigma} < 1-(j-1)\rho$ for some $j \in [2:L]$.The supersymbol that we transmit is S_j because after $(j-1)$ slots, the battery level is $\bar{\sigma}+(j-1)\rho \geqslant 1-\rho$. Note that $\bar{\sigma}+(j-1)\rho < 1$: If $\sigma \geqslant 1$, then $\bar{\sigma}+(j-1)\rho < \sigma$, thus we transmitted S_j without energy overflow; otherwise, if $\sigma < 1$, an energy overflow may have occurred. In the former case (no energy overflow), the energy consumed by each supersymbol is equal to $1-L\rho$, thus we can transmit at most $i_{\max} := \sigma'/(1-L\rho)$ supersymbols. In the latter case (energy overflow), we can transmit less than $\sigma'/(1-L\rho)$ supersymbols, and in particular, we can transmit at most $i_{\max} := \sigma'/\chi$ supersymbols, where χ is an upper bound on the difference between the battery level after and before the transmission of the supersymbol given by $\chi \leqslant 1 - L\rho + \rho = 1-(L-1)\rho$. Indeed, as $L\rho < 1$, there is at most one energy overflow occurring in the battery during a supersymbol transmission, hence the energy overflowed is less than ρ. Writing jointly the two cases yields the statement. □

In a third achievable scheme, we allow transmission of either symbol $\texttt{0}$ or length-L supersymbol $\texttt{00}\cdots\texttt{01}$ satisfying $L \geqslant 1/\rho$ (cf. Fig. 4). The particularity of this scheme is that, as $L\rho \geqslant 1$, it allows to use an infinite number of supersymbols regardless of the battery level before transmission. The resulting lower bound on $v(\sigma,\rho)$ is independent of $\sigma \geqslant 1-\rho$.

Theorem 7. *Given $\sigma \geqslant 1-\rho$. For $L \geqslant 1/\rho$, the exponent of $|\mathcal{S}_n(\sigma,\rho)|$ satisfies*

$$v(\sigma,\rho) \geqslant \max_{0<\tilde{\alpha}\leqslant 1-1/L} (1-\tilde{\alpha})H_2\left(\frac{\tilde{\alpha}}{1-\tilde{\alpha}}\cdot\frac{1}{L-1}\right). \tag{31}$$

Proof. Let i_0 and i_1 be the number of symbol $\texttt{0}$ and length-L supersymbols transmitted. The signaling scheme is as follows: We start transmission with a number of $\texttt{0}$s equal to $\lceil\sigma/\rho\rceil$ and then transmit i_0 symbols $\texttt{0}$ and i_1 supersymbols according to all possible combinations, which are $\binom{i_0+i_1}{i_1}$. The necessary number of time slots is $i_0 + Li_1 =: k = n - \lceil\sigma/\rho\rceil$ (cf. Fig. 4), hence the following exponent is achievable:

$$v(\sigma,\rho) \overset{\text{(a)}}{=} \lim_{n\to\infty}\frac{1}{n}\log_2|\mathcal{S}_n(\sigma,\rho)|$$

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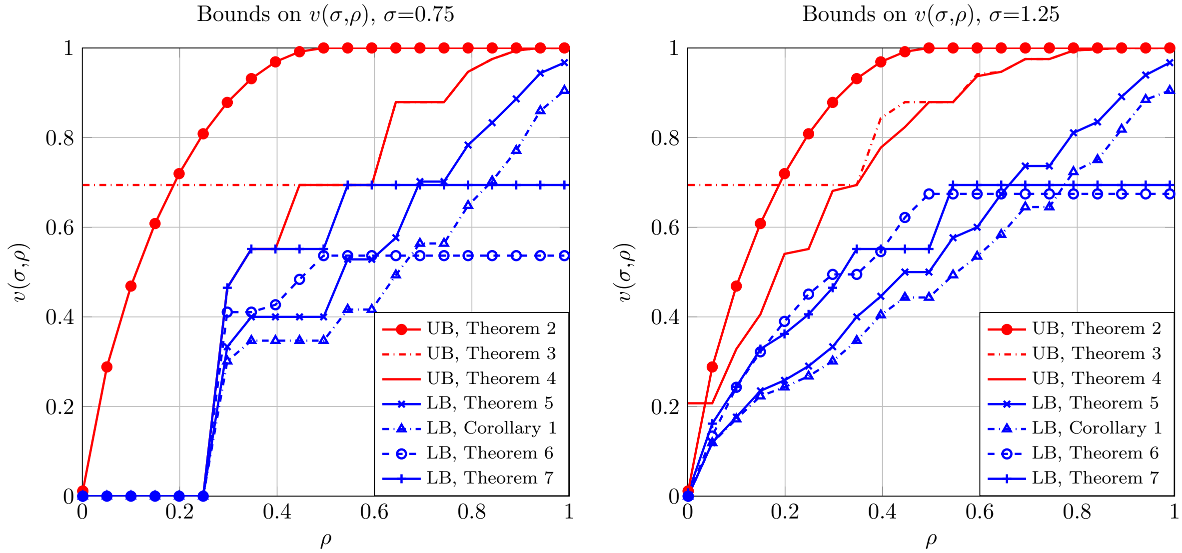
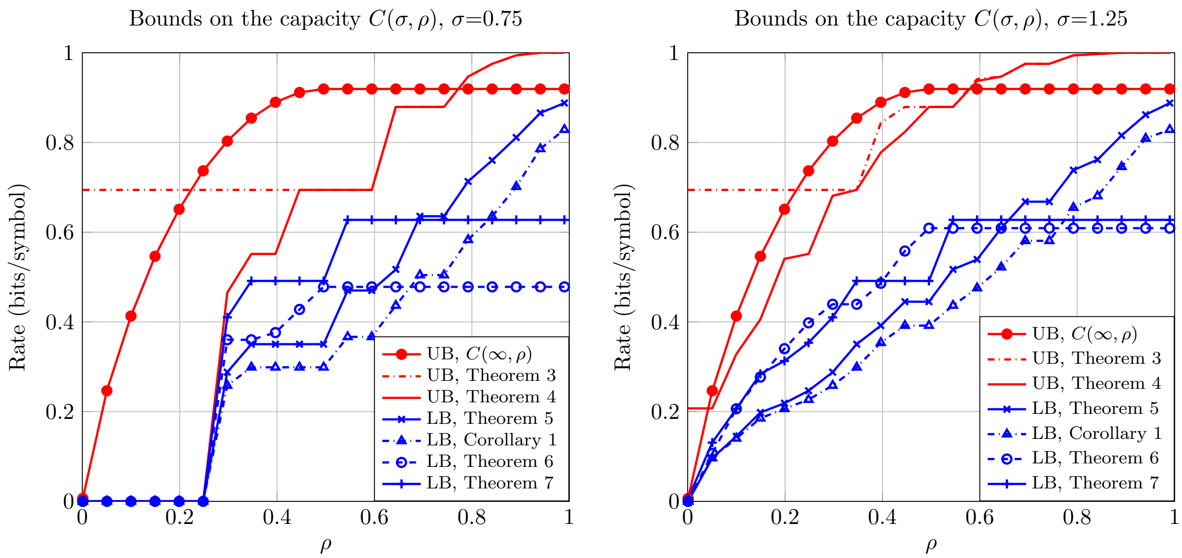


(a) (b)

Fig. 5. Bounds on the exponent of $|\mathcal{S}_n(\sigma,\rho)|$, $v(\sigma,\rho)$, as a function of ρ, for different values of σ.



(a) (b)

Fig. 6. Bounds on the channel capacity of EHBSC, $C(\sigma,\rho)$, as a function of ρ, for two values of σ. In both cases, the crossover probability is $q = 0.01$.

$$\begin{aligned}
&\overset{\text{(b)}}{\geqslant} \lim_{k\to\infty} \frac{1}{k+\lceil\sigma/\rho\rceil}\log_2\binom{k-i_1(L-1)}{i_1} \\
&\overset{\text{(c)}}{\geqslant} \lim_{k\to\infty}\left[\frac{k-i_1(L-1)}{k+\lceil\sigma/\rho\rceil}H_2\left(\frac{i_1}{k-i_1(L-1)}\right)\right. \\
&\qquad\qquad\qquad\left. -\frac{\log_2[k+1-i_1(L-1)]}{k+\lceil\sigma/\rho\rceil}\right] \\
&\overset{\text{(d)}}{=} (1-\tilde{\alpha})H_2\left(\frac{\tilde{\alpha}}{1-\tilde{\alpha}}\cdot\frac{1}{L-1}\right)
\end{aligned}$$

where: (a) is by definition; (b) is by counting the number of possible sequences that we can transmit according to the adopted achievable scheme; (c) follows from lower bounding the binomial; and (d) is by setting $\tilde{\alpha} = i_1(L-1)/k$ and by letting $k\to\infty$. Since $i_1 \leqslant k/L$ by construction, the result is obtained by maximizing over $0 < \tilde{\alpha} < 1-1/L$. $\square$

VI. Numerical results

Numerical results are presented for several values of battery size σ, energy arrival rate ρ, and crossover probability q. Figures 5(a) and 5(b) show the exponent $v(\sigma,\rho)$ as a function of ρ for $\sigma = 0.75$ and $\sigma = 1.25$, respectively. Figures 6(a) and 6(b) show upper and lower bounds on capacity as a function of ρ for the same values of battery size as Figs. 5(a) and 5(b), respectively. Some of the bounds are

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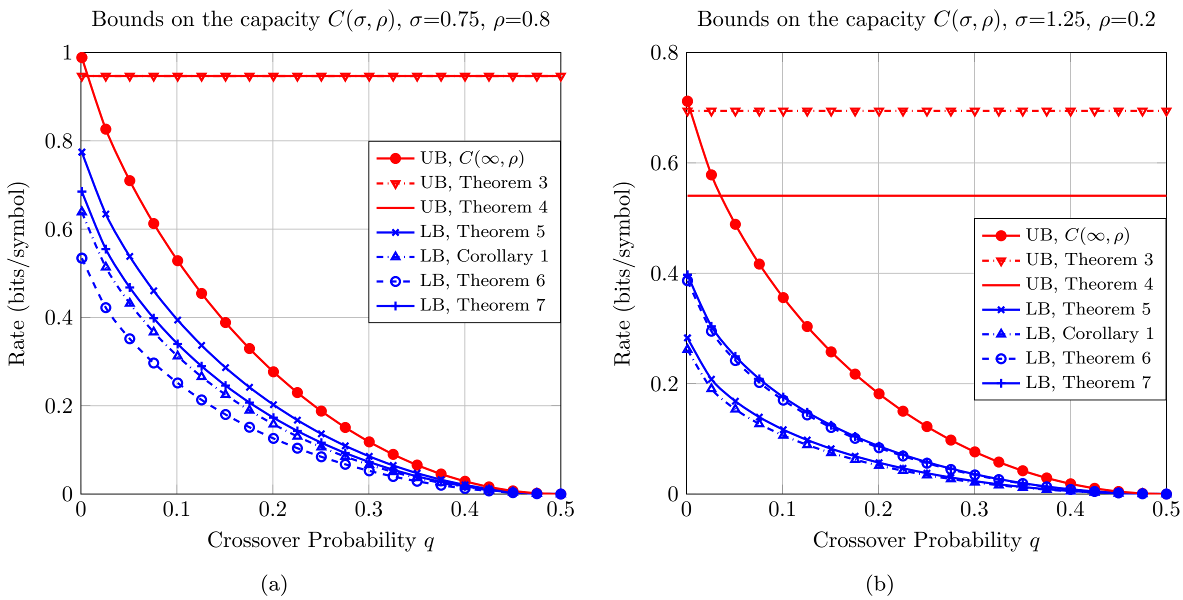

Fig. 7. Bounds on the channel capacity of EHBSC, $C(\sigma,\rho)$, as a function of q, for different values of (σ,ρ).

stair functions because of the presence of floor and ceiling functions in their mathematical expressions. Bounds derived in Theorem 4, 5 and 6 are depicted according to their numerical simulations, which imply suboptimal solutions of the optimization problems. Therefore, curves referring to these theorems may be conservative.

In Fig. 5(a), the exponent is zero for $\rho < 0.25$ and consequently $C(\sigma,\rho) = 0$ (cf. Theorem 1) as shown in Fig. 6(a). Indeed, for such low energy arrival, it leads to $\sigma < 1-\rho$ and $C(\sigma,\rho) = 0$ (cf. III-A2). It is shown that the bounds are particularly close for low ($\rho < 0.4$) and high ($\rho > 0.95$) energy arrival rates. The best lower bounds are provided by either Theorem 5 or 7. The best upper bound is provided by either Theorem 1 or 4. For $\rho \geqslant 0.45$, the upper bound on $v(0.75,\rho)$ given in Theorem 3 is coincident with that in Theorem 4. This manifests that for these large ρ cases, the (l_1, w_{l_1})-constraint becomes the most sensitive one among all (l, w_l)-constraints and Theorem 3 provides simple and effective upper bound. The behavior of both $v(\sigma,\rho)$ and $C(\sigma,\rho)$ for low energy arrival rate ρ is different when $\sigma > 1$. As shown in Figs. 5(b) and 6(b), it is possible to reliably communicate at nonzero rate for any $\rho > 0$. Differently from Figs. 5(a) and 6(a), there are intervals of ρ where the best lower bound is provided by Theorem 6. The best upper bound is provided by either Theorem 1 or 4. For high $\rho > 0.5$, the bound given in Theorem 3 is close to that given in Theorem 4, which can be computed only approximately up to some large value of $l \geqslant 1$ (cf. (20)). Conversely, bound in Theorem 3 is analytical and quickly computable, and furthermore they can be regarded as accurate approximations of Theorem 4 for these high ρ cases.

Finally, Figs. 7(a) and 7(b) show capacity bounds as a function of the crossover probability q for fixed σ and ρ pairs equal to $(0.75, 0.8)$ and $(1.25, 0.2)$, respectively. It is shown that lower bounds are close to the capacity of a system with infinite battery even for small battery size, i.e., $\sigma < \rho$ (cf. Fig. 7(a)), and low arrival rate, i.e., $\rho < \sigma$ (cf. Fig. 7(b)).

VII. Conclusion

We investigated the capacity of energy harvesting binary symmetric channels with (σ,ρ)-power constraint, where σ is the size of the battery and ρ is the amount of energy harvested per slot. We derived upper and lower bounds on capacity in terms of the normalized exponent of the cardinality of the set of feasible input sequences. Upper bounds were derived by relaxing some of the constraints imposed by the harvesting process on the evolution of the battery state. Lower bounds were derived by applying the binary entropy-power inequality and lower bounding the exponent by using specific signaling schemes based on the save-and-transmit strategy. Numerical results for several values of σ and ρ showed that the bounds are close even when the battery size is small, and provided evidence of the effectiveness of the save-and-transmit strategy or its variations. The established bounds on the channel with (σ,ρ)-power constraint can be used to optimize the performance of energy harvesting communications systems from a higher layer perspective where the channel is modeled as binary. It is valuable to extend these bounds to the multiuser setting, e.g. multiple access channels and broadcast channels, with many users, which can form the basis for a deeper understanding of IoT networks.

Acknowledgment

We would like to thank the editor and the anonymous reviewers for their insightful comments and suggestions, which are very helpful in improving the quality of this paper.

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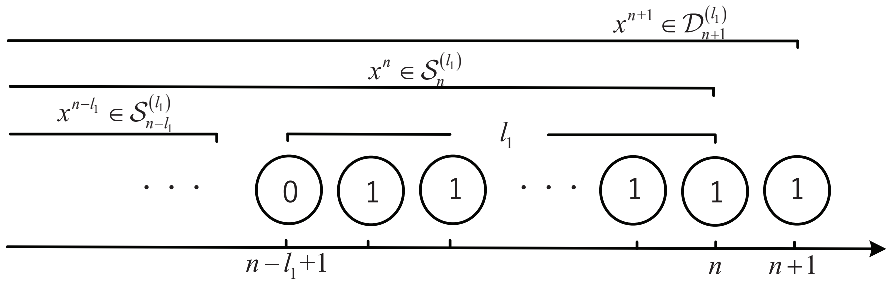

Fig. 8. Generic sequence $x^{n+1} \in \mathcal{D}_{n+1}^{(l_1)}$ and its relation to sequences in $\mathcal{P}_{n+1}^{(l_1)}$ that violates the Hamming weight $l_1 - 1$ constraint in length-l_1 window.

Appendix

A. Proof of Theorem 3

We divide the proof into two subsections which upper bound $v(\sigma, \rho)$ by the (normalized) exponent of $|\mathcal{S}_n^{(l_1)}|$ and that of $|\mathcal{S}_n^{(l_2)}|$, respectively, since $|\mathcal{S}_n| \leqslant |\mathcal{S}_n^{(l)}|$ for any $l \in [1:n]$.

1) In this subsection, we bound $v(\sigma, \rho)$ by studying $|\mathcal{S}_n^{(l_1)}|$. Recall that

$$l_1 - 1 \leqslant w_{l_1} < l_1.$$

The Hamming weight of each length-l_1 window in $\mathcal{S}_n^{(l_1)}$ satisfies

$$\mathsf{w}(x_{i+1}^{i+l_1}) \leqslant w_{l_1} < l_1, \quad i \in [0 : n - l_1].$$

Note that for each sequence $x^{n+1} \in \mathcal{S}_{n+1}^{(l_1)}$, the subsequence of leading n symbols in x^{n+1}, i.e., x_1^n, belongs to $\mathcal{S}_n^{(l_1)}$, because the set of constraint in $\mathcal{S}_{n+1}^{(l_1)}$ includes those defining $\mathcal{S}_n^{(l_1)}$. Define the set of padded sequences as follows:

$$\mathcal{P}_{n+1}^{(l_1)} := \{x^{n+1} \in \mathbb{Z}_2^{n+1} : \; x_1^n \in \mathcal{S}_n^{(l_1)}, \; x_{n+1} \in \mathbb{Z}_2\}.$$

In other words, $x^{n+1} \in \mathcal{P}_{n+1}^{(l)}$ is generated from a sequence in $\mathcal{S}_n^{(l_1)}$, at the end of which we pad either a $\mathsf{0}$ or a $\mathsf{1}$. Hence, $|\mathcal{P}_{n+1}^{(l_1)}| = 2|\mathcal{S}_n^{(l_1)}|$. Not all sequences in the padded set are feasible, i.e., $\mathcal{D}_{n+1}^{(l_1)} := \mathcal{P}_{n+1}^{(l_1)} \backslash \mathcal{S}_{n+1}^{(l_1)} \neq \emptyset$. We are interested in the cardinality of the difference set:

$$|\mathcal{D}_{n+1}^{(l_1)}| = |\mathcal{P}_{n+1}^{(l_1)}| - |\mathcal{S}_{n+1}^{(l_1)}|. \tag{32}$$

In order to compute $|\mathcal{S}_{n+1}^{(l_1)}|$, it remains to evaluate $|\mathcal{D}_{n+1}^{(l_1)}|$. From the definition of $\mathcal{D}_{n+1}^{(l_1)}$, we can deduce that

$$x^{n+1} \in \mathcal{D}_{n+1}^{(l_1)} \implies \mathsf{w}(x_{n+1-(l_1-1)}^{n+1}) = l_1. \tag{33}$$

This is so because the (l_1, w_{l_1})-constraint is satisfied up to the last symbol of all sequences in the padded set, i.e.,

$$x^{n+1} \in \mathcal{P}_{n+1}^{(l_1)} \implies \mathsf{w}(x_{i+1}^{i+l_1}) \leqslant w_{l_1}, \;\; i + l_1 \leqslant n.$$

Since

$$\mathcal{S}_{n+1}^{(l_1)} = \{x^{n+1} \in \mathbb{Z}_2^{n+1} : \mathsf{w}(x_{i+1}^{i+l_1}) \leqslant w_{l_1}, \;\; i + l_1 \leqslant n+1\},$$

then

$$x^{n+1} \in \mathcal{D}_{n+1}^{(l_1)} \implies \mathsf{w}(x_{i+1}^{i+l_1}) > w_{l_1} \text{ for } i + l_1 = n+1.$$

Therefore, the subsequence of l_1 trailing symbols in $x^{n+1} \in \mathcal{D}_{n+1}^{(l_1)}$ does not satisfy the (l_1, w_{l_1})-constraint. On the other hand, any subsequence of length l_1 cannot have Hamming weight larger than l_1. Thus

$$x^{n+1} \in \mathcal{D}_{n+1}^{(l_1)} \implies l_1 - 1 \leqslant w_{l_1} < \mathsf{w}(x_{n+1-(l_1-1)}^{n+1}) \leqslant l_1.$$

So we proved (33). Hence we also know that the l_1 trailing symbols are $\mathsf{1}$s (cf. Fig. 8). This implies $x_{n-l_1+1} = \mathsf{0}$ because $x_1^n \in \mathcal{S}_n^{(l_1)}$, thus the (l_1, w_{l_1})-constraint is satisfied, but at the same time, the $l_1 - 1$ trailing symbols of x_1^n are $\mathsf{1}$s. Therefore, $\mathcal{D}_{n+1}^{(l_1)}$ can be explicitly characterized as follows:

$$\begin{aligned}\mathcal{D}_{n+1}^{(l_1)} = \{x^{n+1} \in \mathbb{Z}_2^{n+1} : \; & x^{n-l_1} \in \mathcal{S}_{n-l_1}^{(l_1)}, \\ & x_{n-l_1+1} = \mathsf{0}, \; x_{n-l_1} = \cdots = x_{n+1} = \mathsf{1}\}.\end{aligned}$$

Accordingly, $|\mathcal{D}_{n+1}^{(l_1)}| = |\mathcal{S}_{n-l_1-1}^{(l_1)}|$. Following from (32), for $n > l_1$, we have

$$|\mathcal{S}_{n+1}^{(l_1)}| = 2|\mathcal{S}_n^{(l_1)}| - |\mathcal{S}_{n-l_1}^{(l_1)}|. \tag{34}$$

Defining $d_n := |\mathcal{S}_n^{(l_1)}|$ allows to derive from (34) the following recurrence equation:

$$d_n = 2d_{n-1} - d_{n-l_1-1}, \;\; n > l_1, \tag{35}$$

$$d_n = 2^n, \;\; n \in [0, l_1 - 1], \tag{36}$$

$$d_{l_1} = 2^{l_1} - 1. \tag{37}$$

We are interested in the asymptotic growth of d_n. The approach that we use relies on generating functions. Define

$$D(z) := \sum_{n \geqslant 0} d_n z^n.$$

Summing (35) over $n \geqslant l_1 + 1$ yields

$$\left(D(z) - \sum_{n=0}^{l_1} d_n z^n\right) - 2z\left(D(z) - \sum_{n=0}^{l_1-1} d_n z^n\right) + z^{l_1+1} D(z) = 0.$$

Rearranging and using (36)-(37) yields

$$D(z) = \frac{P_1(z)}{Q_1(z)}, \;\; P_1(z) := 1 - z^{l_1}, \;\; Q_1(z) := 1 - 2z + z^{l_1+1}.$$

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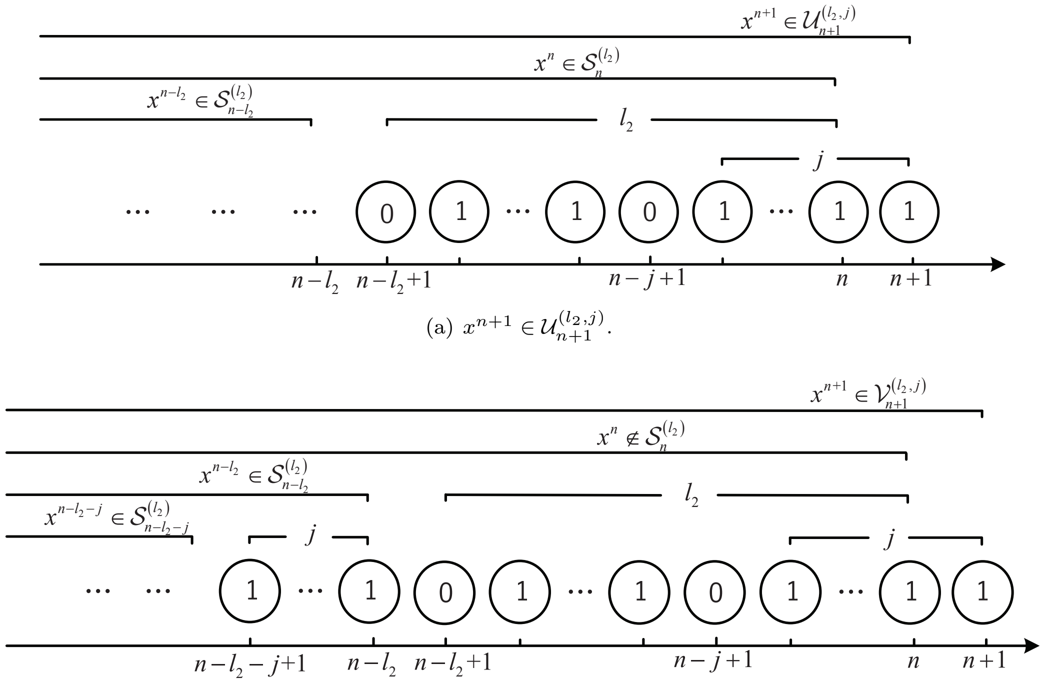

(a) $x^{n+1} \in \mathcal{U}_{n+1}^{(l_2,j)}$.

(b) $x^{n+1} \in \mathcal{V}_{n+1}^{(l_2,j)}$.

Fig. 9. Generic sequences $x^{n+1} \in \mathcal{U}_{n+1}^{(l_2,j)}$ and $x^{n+1} \in \mathcal{V}_{n+1}^{(l_2,j)}$. In particular, $x^{n+1} \in \mathcal{U}_{n+1}^{(l_2,j)} \backslash \mathcal{V}_{n+1}^{(l_2)}$ violates the (l_2, w_{l_2})-constraint for length-l_2 window ending at symbol $x_{n-l_2+2}, x_{n-l_2+3}, \cdots, x_{n-j}$.

It is known (cf. [35, Theorem 4.1]) that $|\mathcal{S}_n^{(l_1)}| = d_n$ grows as $C_1 \beta_D^n n^{\nu_1 - 1}$, where C_1 is a constant, $1/\beta_D$ is the pole of $D(z)$ with smallest modulus, and ν_1 is the multiplicity of the pole. Denote $1/\beta_1$ the zero of $Q_1(z)$ with smallest modulus. Then, $1/\beta_D \geqslant 1/\beta_1$ and we can bound $v(\sigma, \rho)$ as follows:

$$\begin{aligned} v(\sigma,\rho) &= \lim_{n\to\infty} \frac{1}{n} \log_2 |\mathcal{S}_n(\sigma,\rho)| \\ &\leqslant \lim_{n\to\infty} \frac{1}{n} \log_2 |\mathcal{S}_n^{(l_1)}| \\ &= \lim_{n\to\infty} \frac{1}{n} \log_2 (C_1 \beta_D^n n^{\nu_1 - 1}) \\ &\leqslant \log_2 \beta_1. \end{aligned} \tag{38}$$

2) In this subsection, we bound $v(\sigma, \rho)$ by studying $|\mathcal{S}_n^{(l_2)}|$. Based on the fact

$$l_2 - 2 \leqslant w_{l_2} < l_2 - 1,$$

the Hamming weight of each length-l_2 window in $\mathcal{S}_{n+1}^{(l_2)}$ is lower than $l_2 - 1$, i.e.

$$\mathsf{w}(x_{i+1}^{i+l_2}) \leqslant w_{l_2} < l_2 - 1, \quad i \in [0 : n - l_2].$$

In order to derive an upper bound on $|\mathcal{S}_{n+1}^{(l_2)}|$, we define $\mathcal{P}_{n+1}^{(l_2)} \supseteq \mathcal{S}_{n+1}^{(l_2)}$ and $\mathcal{D}_{n+1}^{(l_2)} := \mathcal{P}_{n+1}^{(l_2)} \backslash \mathcal{S}_{n+1}^{(l_2)}$, and we aim to lower bound $|\mathcal{D}_{n+1}^{(l_2)}|$. Let $\mathcal{P}_{n+1}^{(l_2)}$ be the following set:

$$\mathcal{P}_{n+1}^{(l_2)} := \{x^{n+1} \in \mathbb{Z}_2^{n+1} : \ x_1^n \in \mathcal{S}_n^{(l_2)}, \ x_{n+1} \in \mathbb{Z}_2\}.$$

Therefore, $|\mathcal{P}_{n+1}^{(l_2)}| = 2|\mathcal{S}_n^{(l_2)}|$. We divide the proof in steps where a lower bound on $\mathcal{D}_{n+1}^{(l_2)}$ is derived in Step 1–3, and an upper bound on $|\mathcal{S}_{n+1}^{(l_2)}|$ is derived in Step 4.

Step 1. We prove that

$$x^{n+1} \in \mathcal{D}_{n+1}^{(l_2)} \implies \mathsf{w}(x_{n+1-(l_2-1)}^{n+1}) = l_2 - 1. \tag{39}$$

Indeed, by definition of $\mathcal{D}_{n+1}^{(l_2)}$, the (l_2, w_{l_2})-constraint is satisfied for all length-l_2 subsequences of x_1^n, but $x^{n+1} \notin \mathcal{S}_{n+1}^{(l_2)}$, thus the l_2 trailing symbols of x^{n+1} do not satisfy the (l_2, w_{l_2})-constraint: $\mathsf{w}(x_{n+1-(l_2-1)}^{n+1}) > w_{l_2} \geqslant l_2 - 2$. Since $x_1^n \in \mathcal{S}_n^{(l_2)}$, we also have $\mathsf{w}(x_{n-(l_2-1)}^n) \leqslant w_{l_2} < l_2 - 1$, which implies $\mathsf{w}(x_{n-(l_2-1)}^{n+1}) < l_2$. Thus we have $l_2 - 2 < \mathsf{w}(x_{n+1-(l_2-1)}^{n+1}) < l_2$.

Step 2. We prove that

$$x^{n+1} \in \mathcal{D}_{n+1}^{(l_2)} \implies x_{n+1} = \mathtt{1} \text{ and } x_{n-l_2+1} = \mathtt{0}. \tag{40}$$

Indeed, since $\mathsf{w}(x_{n+1-(l_2-1)}^{n+1}) = l_2 - 1$ from Step 1, there is just one $\mathtt{0}$ among the l_2 trailing symbols of x^{n+1}. On the other hand, $x_{n+1} = \mathtt{1}$ because $x_1^n \in \mathcal{S}_n^{(l_2)}$ and if x_{n+1} were $\mathtt{0}$, we would have $x^{n+1} \in \mathcal{S}_{n+1}^{(l_2)}$, which leads to a contradiction. This also implies that $\mathsf{w}(x_{n+1-(l_2-1)}^n) = l_2 - 2$. We prove that $x_{n-l_2+1} = \mathtt{1}$ leads to a contradiction. Assuming $x_{n-l_2+1} = \mathtt{1}$ implies $\mathsf{w}(x_{n-(l_2-1)}^n) = \mathsf{w}(x_{n-l_2+2}^n) + 1 = l_2 - 1$, but since $x_1^n \in \mathcal{S}_n^{(l_2)}$, it has $\mathsf{w}(x_{n-l_2+2}^n) \leqslant w_{l_2} < l_2 - 1$.

Step 1 and 2 imply that the general structure of the $l_2 + 1$ trailing symbols of $x^{n+1} \in \mathcal{D}_{n+1}^{(l_2)}$ is as follows:

$$\underbrace{\mathtt{0}\ \mathtt{1} \cdots \mathtt{1}\ \mathtt{0}\ \underbrace{\mathtt{1} \cdots \mathtt{1}}_{j}}_{l_2+1}, \tag{41}$$

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with $j \in [1 : l_2 - 1]$. We define $\mathcal{D}_{n+1}^{(l_2,j)}$ as the subset of $\mathcal{D}_{n+1}^{(l_2)}$ having the $l_2 + 1$ trailing symbols as described in (41): these sets represent a partition of $\mathcal{D}_{n+1}^{(l_2)}$.

Step 3. We define $l_2 - 1$ pairs of sets $\{\mathcal{U}_{n+1}^{(l_2,j)}, \mathcal{V}_{n+1}^{(l_2,j)}\}$, each pair corresponding to one configuration of the $l_2 + 1$ trailing symbols of $x^{n+1} \in \mathcal{D}_{n+1}^{(l_2)}$, that are useful to lower bound the cardinality of $\mathcal{D}_{n+1}^{(l_2,j)}$ as follows:

$$|\mathcal{D}_{n+1}^{(l_2,j)}| = |\mathcal{U}_{n+1}^{(l_2,j)}| - |\mathcal{U}_{n+1}^{(l_2,j)} \backslash \mathcal{D}_{n+1}^{(l_2,j)}| \geqslant |\mathcal{U}_{n+1}^{(l_2,j)}| - |\mathcal{V}_{n+1}^{(l_2,j)}|. \tag{42}$$

These sets are formally defined as follows (cf. Fig. 9):

$$\begin{aligned}
\mathcal{U}_{n+1}^{(l_2,j)} := \{x^{n+1} \in \mathbb{Z}_2^{n+1} :\ & x_1^{n-l_2} \in \mathcal{S}_{n-l_2}^{(l_2)}, & (43)\\
& x_{n-l_2+1} = x_{n-j+1} = \mathtt{0}, & (44)\\
& x_{n-l_2+2} = x_{n-l_2+3} = \cdots = x_{n-j} = \mathtt{1}, & (45)\\
& x_{n-j+2} = x_{n-j+3} = \cdots = x_{n+1} = \mathtt{1}\}, & (46)\\
\mathcal{V}_{n+1}^{(l_2,j)} := \{x^{n+1} \in \mathbb{Z}_2^{n+1} :\ & x_1^{n-l_2-j} \in \mathcal{S}_{n-l_2-j}^{(l_2)}, & (47)\\
& x_{n-l_2-j+1} = x_{n-l_2-j+2} = \cdots = x_{n-l_2-1} = \mathtt{1}, & (48)\\
& x_{n-l_2+1} = x_{n-j+1} = \mathtt{0}, & (49)\\
& x_{n-l_2+2} = x_{n-l_2+3} = \cdots = x_{n-j} = \mathtt{1}, & (50)\\
& x_{n-j+2} = x_{n-j+3} = \cdots = x_{n+1} = \mathtt{1}\}. & (51)
\end{aligned}$$

In other words, (44)–(46) and (49)–(51) formally define the structure in (41). Hence, the $l_2 + 1$ trailing symbols of sequences in $\mathcal{U}_{n+1}^{(l_2,j)}$ and $\mathcal{V}_{n+1}^{(l_2,j)}$ are the same as in $\mathcal{D}_{n+1}^{(l_2,j)}$. The difference between $\mathcal{U}_{n+1}^{(l_2,j)}$ and $\mathcal{D}_{n+1}^{(l_2,j)}$ lies in the leading $n - l_2$ symbols: in particular, sequences in $\mathcal{U}_{n+1}^{(l_2,j)}$ have a length-$(n - l_2)$ prefix drawn from $\mathcal{S}_{n-l_2}^{(l_2)}$. This makes $\mathcal{U}_{n+1}^{(l_2,j)}$ a superset of $\mathcal{D}_{n+1}^{(l_2,j)}$ because some of the sequences do not satisfy the (l_2, w_{l_2})-constraint. To justify the definition of $\mathcal{V}_{n+1}^{(l_2,j)}$, let us note that sequences in $\mathcal{D}_{n+1}^{(l_2,j)}$ satisfy the (l_2, w_{l_2})-constraint up to symbol n, while sequences in $\mathcal{U}_{n+1}^{(l_2,j)}$ satisfy the constraints up to symbol $n - l_2$. Therefore, in order to understand the elements of $\mathcal{U}_{n+1}^{(l_2,j)} \backslash \mathcal{D}_{n+1}^{(l_2,j)}$, we can just check whether the (l_2, w_{l_2})-constraint is not satisfied for sequences ending at $n-l_2+1$, $n-l_2+2$, ..., n. Since one the (l_2, w_{l_2})-constraints is not satisfied, then $x_{n-l_2-j+1} = \cdots = x_{n-l_2} = \mathtt{1}$. Therefore, the structure of the trailing l_2+j+1 symbols of sequences in $\mathcal{U}_{n+1}^{(l_2,j)} \backslash \mathcal{D}_{n+1}^{(l_2,j)}$ is as follows:

$$\underbrace{\underbrace{\mathtt{1}\cdots\mathtt{1}}_{j}\ \mathtt{0}\ \mathtt{1}\cdots\mathtt{1}\ \mathtt{0}\ \underbrace{\mathtt{1}\cdots\mathtt{1}}_{j}}_{l_2+j+1}\ . \tag{52}$$

The subsequences $x_1^{n-l_2-j}$ of sequences in $\mathcal{U}_{n+1}^{(l_2,j)} \backslash \mathcal{D}_{n+1}^{(l_2,j)}$ belong to $\mathcal{S}_{n-l_2-j}^{(l_2)}$, but not all sequences in $\mathcal{S}_{n-l_2-j}^{(l_2)}$ can be found as subsequences $x_1^{n-l_2-j}$ of sequences in $\mathcal{U}_{n+1}^{(l_2,j)} \backslash \mathcal{D}_{n+1}^{(l_2,j)}$. Therefore, $\mathcal{V}_{n+1}^{(l_2,j)}$ is a superset of $\mathcal{U}_{n+1}^{(l_2,j)} \backslash \mathcal{D}_{n+1}^{(l_2,j)}$. Finally, we notice that $|\mathcal{U}_{n+1}^{(l_2,j)}| = |\mathcal{S}_{n-l_2}^{(l_2)}|$ and $|\mathcal{V}_{n+1}^{(l_2,j)}| = |\mathcal{S}_{n-l_2-j}^{(l_2)}|$. Hence, from (42) and being $\{\mathcal{D}_{n+1}^{(l_2,j)}, j \in [1 : l_2 - 1]\}$ a partition of $\mathcal{D}_{n+1}^{(l_2)}$, we have

$$|\mathcal{D}_{n+1}^{(l_2)}| = \sum_{j=1}^{l_2-1} |\mathcal{D}_{n+1}^{(l_2,j)}| \geqslant \sum_{j=1}^{l_2-1} |\mathcal{S}_{n-l_2}^{(l_2)}| - |\mathcal{S}_{n-l_2-j}^{(l_2)}|.$$

Step 4. An upper bound on $|\mathcal{S}_{n+1}^{(l_2)}|$ is as follows:

$$\begin{aligned}
|\mathcal{S}_{n+1}^{(l_2)}| &= |\mathcal{P}_{n+1}^{(l_2)}| - |\mathcal{D}_{n+1}^{(l_2)}|\\
&\geqslant 2|\mathcal{S}_{n}^{(l_2)}| - \sum_{j=1}^{l_2-1} (|\mathcal{S}_{n-l_2}^{(l_2)}| - |\mathcal{S}_{n-l_2-j}^{(l_2)}|)\\
&= 2|\mathcal{S}_{n}^{(l_2)}| - (l_2 - 1)|\mathcal{S}_{n-l_2}^{(l_2)}| + \sum_{j=1}^{l_2-1} |\mathcal{S}_{n-l_2-j}^{(l_2)}|. \qquad (53)
\end{aligned}$$

Since we aim to upper bound the growth exponent of $|\mathcal{S}_n^{(l_2)}|$ as $n \to \infty$, we consider a sequence a_n which grows faster than $|\mathcal{S}_n^{(l_2)}|$:

$$a_{n+1} = 2a_n - (l_2 - 1)a_{n-l_2} + \sum_{j=1}^{l_2-1} a_{n-l_2-j}.$$

By applying the method of generating function as in Appendix A-1), we define $A(z) := \sum_{n\geqslant 0} a_n z^n$ and find its pole $1/\beta_A$ of smallest modulus. Denote $A(z) = P_2(z)/Q_2(z)$. Then we have

$$Q_2(z) = 1 - 2z + (l_2 - 1)z^{l_2+1} - \sum_{j=1}^{l_2-1} z^{l_2+1+j}. \tag{54}$$

The sequence a_n grows asymptotically as $C_2 \beta_A^n n^{\nu-1}$, where C_2 is a constant and ν_2 is the multiplicity of the pole at $1/\beta_A$. Denote $1/\beta_2$ the zero of $Q_2(z)$ with smallest modulus. Then $\beta_2 \geqslant \beta_A$, hence we can bound the exponent of $|\mathcal{S}_n(\sigma, \rho)|$ as follows:

$$\begin{aligned}
v(\sigma, \rho) &= \lim_{n\to\infty} \frac{1}{n} \log_2 |\mathcal{S}_n(\sigma, \rho)|\\
&\leqslant \lim_{n\to\infty} \frac{1}{n} \log_2 |\mathcal{S}_n^{(l_2)}|\\
&\leqslant \lim_{n\to\infty} \frac{1}{n} \log_2 (C_2 \beta_A^n n^{\nu_2-1})\\
&\leqslant \log_2 \beta_2. \qquad (55)
\end{aligned}$$

Finally, combining (38) and (55) results in the bound stated in Theorem 3.

B. Derivation of (24)

As l_1 satisfies that $l_1 - 1 \leqslant w_{l_1} < l_1$, the length-$l_1$ sequence of all 1s, i.e., $\mathbf{1}_{l_1}$ is the one and the only length-l_1 sequence that violates the (l_1, w_{l_1})-constraint. Hence, for $x^n \in \mathcal{S}_n^{(l_1)}$, $x^{n+1} \notin \mathcal{S}_{n+1}^{(l_1)}$ if $x_{n+1} = \mathtt{1}$ and $x_{n-l_1+2}^n = \mathtt{1}_{l_1-1}$, i.e., the symbol 1 is padded at x_{n+1} when the l_1-1 trailing symbol of x^n are all 1s. In general, we can compactly write $\boldsymbol{A}_{l_1}$ as

$$\boldsymbol{A}_{l_1} := \left[\begin{array}{c|c} \boldsymbol{B}_{l_1} & \boldsymbol{B}_{l_1} \\ \hline \begin{array}{r} 1 \\ 0 \end{array} & \begin{array}{r} 0 \\ 0 \end{array} \end{array}\right], \tag{56}$$

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where

$$\boldsymbol{B}_{l_1} := \boldsymbol{I}_{2^{l_1-1}-1} \otimes \boldsymbol{1}_2 = \begin{bmatrix} \boldsymbol{1}_2 & & \\ & \ddots & \\ & & \boldsymbol{1}_2 \end{bmatrix}, \ \boldsymbol{1}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad (57)$$

Denote the characteristic polynomial of $\boldsymbol{A}_{l_1}$ by $F(\lambda)$. To compute $F(\lambda)$, let us first consider the example of $l_1 = 3$. When $l_1 = 3$,

$$\begin{aligned} F(\lambda) =& |\boldsymbol{A}_{l_1} - \lambda \boldsymbol{I}_{2^{l_1}}| \\ =& \begin{vmatrix} 1-\lambda & & & & 1 & & & \\ 1 & -\lambda & & & 1 & & & \\ & 1 & -\lambda & & & 1 & & \\ & 1 & & -\lambda & & 1 & & \\ & & 1 & & -\lambda & & 1 & \\ & & 1 & & & -\lambda & 1 & \\ & & & 1 & & & -\lambda & 0 \\ & & & 0 & & & & -\lambda \end{vmatrix} \\ =& (-\lambda)^4 \begin{vmatrix} 1-\lambda & & 1 & \\ 1 & -\lambda & 1 & \\ & 1 & -\lambda & 1 \\ & 1 & & -\lambda \end{vmatrix} \\ =& (-\lambda)^5(-\lambda-1) \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda + \frac{\lambda}{\lambda+1} \end{vmatrix} \\ =& (-\lambda)^5(-\lambda-1) \times \\ & \left(-\lambda - 1 + \frac{\lambda}{\lambda+1}\right)\left(1 - \lambda + \frac{\lambda}{\lambda + 1 - \frac{\lambda}{\lambda+1}}\right). \end{aligned}$$

Similar analysis can be extended to the computation of the general form of $F(\lambda)$, which is

$$\begin{aligned} F(\lambda) =& (-\lambda)^{2^{l_1}-l_1} \times \\ & \underbrace{(-\lambda-1)\left(-\lambda-1+\tfrac{\lambda}{\lambda+1}\right)\left(-\lambda-1+\tfrac{\lambda}{\lambda+1-\frac{\lambda}{\lambda+1}}\right) \cdots \left(1-\lambda+\tfrac{\lambda}{\lambda+1-\frac{\lambda}{\lambda+1-\frac{\lambda}{\lambda+1-\cdots}}}\right)}_{l_1 \text{ terms}} \\ \overset{(a)}{=}& \lambda^{2^{l_1}-l_1}\left(\lambda^{l_1} - \sum_{i=0}^{l_1-1}\lambda^i\right) \\ =& \lambda^{2^{l_1}-l_1}\frac{\lambda^{l_1+1} - 2\lambda^{l_1} + 1}{\lambda - 1}, \end{aligned}$$

where (a) follows by calculating the l_1 terms of product in $F(\lambda)$ backwards. Therefore, the largest eigenvalue of $\boldsymbol{A}_{l_1}$, $\lambda_{\max}(\boldsymbol{A}_{l_1})$, is the largest real root of $F(\lambda) = 0$, i.e., the largest real root of

$$\bar{F}(\lambda) := \lambda^{l_1+1} - 2\lambda^{l_1} + 1 = 0. \quad (58)$$

Note that by (16) and $\sigma + \rho \geqslant 1$, $l_1 \geqslant 2$ holds. The first-order derivative of $\bar{F}(\lambda)$, $\bar{F}'(\lambda) = (l_1+1)\lambda^{l_1} - 2l_1\lambda^{l_1-1}$ is positive if $\lambda > 2l_1/(l_1+1)$. The second-order derivative of $\bar{F}(\lambda)$, $\bar{F}''(\lambda) = l_1(l_1+1)\lambda^{l_1-1} - 2l_1(l_1-1)\lambda^{l_1-2}$ is positive if $\lambda > 2(l_1-1)/(l_1+1)$. This manifests that $\bar{F}(\lambda)$ is convex for $\lambda > 2(l_1-1)/(l_1+1)$. Noting that $\bar{F}(2) = 1 > 0$, we can obtain an approximation of the root by applying Newton's method around $\lambda = 2$ and this approximation will be always larger than the actual root as $\bar{F}(\lambda) \geqslant \bar{F}'(2)(\lambda-2) + \bar{F}(2)$. Therefore, since

$$\bar{F}(\lambda) = \bar{F}(2) + \bar{F}'(2)(\lambda-2) + o(\lambda-2) = 1 + 2^{l_1}(\lambda-2) + o(\lambda-2),$$

an upper bound of the largest root of $\bar{F}(\lambda)$ is

$$2 - 2^{-l_1} \geqslant \lambda_{\max}(\boldsymbol{A}_{l_1}).$$

Therefore, the exponent is upper bounded by

$$v(\sigma,\rho) \leqslant \log_2\left(2 - 2^{-l_1}\right) \leqslant 1 - \frac{2^{-l_1-1}}{\log 2}.$$

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